Structure and
Support Vector Machines

Outline

- SVMs for structured outputs
  - Declarative view
  - Procedural view

Notation for Linear Models

- Training data: {(x1, y1), (x2, y2), …, (xN, yN)}
- Testing data: {(xN+1, yN+1), … (xN+N’, yN+N’)}
- Feature function: **g**
- Weights: **w**
- Decoding:

$$\text{decode}(\mathbf{w}, \boldsymbol{x}) = \arg\max_{\boldsymbol{y}} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

- Learning:

$$\text{learn}\left(\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^{N}\right) = \arg\max_{\mathbf{w}} \Phi\left(\mathbf{w}, \{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^{N}\right)$$

- Evaluation:

$$\frac{1}{N'} \sum_{i=1}^{N'} \text{cost}\left(\text{decode}\left(\text{learn}\left(\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^{N}\right), \boldsymbol{x}_{N+i}\right), \boldsymbol{y}_{N+i}\right)$$

Empirical Risk Minimization

- A unifying framework for many learning algorithms.

$$\begin{aligned}
\text{learn}\left(\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^N\right) &= \arg\max_{\mathbf{w}} \Phi\left(\mathbf{w}, \{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^N\right) \\
&= \arg\min_{\mathbf{w}} \underbrace{\frac{1}{N}\sum_{i=1}^{N} L(\mathbf{w}, \boldsymbol{x}_i, \boldsymbol{y}_i)}_{\approx\, \mathbb{E}[L(\mathbf{w}, \boldsymbol{X}, \boldsymbol{Y})]} + R(\mathbf{w})
\end{aligned}$$

- Many options for the loss function L and the regularization function R.

Loss Functions You Know

Name	Expression of $L(\mathbf{w}, x, y)$
Log loss (joint)	$-\log p(\boldsymbol{x}, \boldsymbol{y} \mid \mathbf{w})$
Log loss (conditional)	$-\log p(\boldsymbol{y} \mid \boldsymbol{x}, \mathbf{w})$
Zero-one loss	$\mathbf{1}\{\text{decode}(\mathbf{w}, \boldsymbol{x}) \neq \boldsymbol{y}\}$
Expected zero-one loss	$1 - p(\boldsymbol{y} \mid \boldsymbol{x}, \mathbf{w})$

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Log loss (conditional)	$-\log p(\boldsymbol{y} \mid \boldsymbol{x}, \mathbf{w})$
Cost	$\mathrm{cost}(\mathrm{decode}(\mathbf{w}, \boldsymbol{x}), \boldsymbol{y})$
Expected cost, a.k.a. “risk”	$\mathbb{E}_{p(\boldsymbol{Y} \mid \boldsymbol{x}, \mathbf{w})}[\mathrm{cost}(\boldsymbol{Y}, \boldsymbol{y})]$

Structured Perceptron

- Described as an online algorithm.
- On each iteration, take one example, and update the weights according to:

$$\mathbf{w} \quad \leftarrow \quad \mathbf{w} + \alpha \left(\mathbf{g}(\boldsymbol{x}_t, \boldsymbol{y}_t) - \mathbf{g}(\boldsymbol{x}_t, \mathrm{decode}(\mathbf{w}, \boldsymbol{x}_t)) \right)$$

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Log loss (conditional)	$-\log p(\boldsymbol{y} \mid \boldsymbol{x}, \mathbf{w})$
Cost	$\mathrm{cost}(\mathrm{decode}(\mathbf{w}, \boldsymbol{x}), \boldsymbol{y})$
Expected cost, a.k.a. “risk”	$\mathbb{E}_{p(\boldsymbol{Y} \mid \boldsymbol{x}, \mathbf{w})}[\mathrm{cost}(\boldsymbol{Y}, \boldsymbol{y})]$
Perceptron loss	$-\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}')$

The Ideal Loss Function

- Convex
- Continuous
- Cost-aware

Cost and Margin

- The “margin” is an important concept when we take the linear models point of view.
  - A “large margin” means that the correct output is well-separated from the incorrect outputs.
- Neither log loss nor “perceptron loss” takes into account the *cost* function, though.
  - In other words, some incorrect outputs are worse than others.

Multiclass SVM (Crammer and Singer, 2001)

$$\begin{aligned}
&\max_{\mathbf{w}} \gamma \\
&\quad \text{s.t.} \quad \|\mathbf{w}\| \leq 1 \\
&\qquad\qquad \forall i, \forall \boldsymbol{y},\ \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) \geq \begin{cases} \gamma & \text{if } \boldsymbol{y} \neq \boldsymbol{y_i} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

- The above can be understood as a 0-1 cost; let's generalize a bit:

$$\begin{aligned}
&\max_{\mathbf{w}} \gamma \\
&\quad \text{s.t.} \quad \|\mathbf{w}\| \leq 1 \\
&\qquad\qquad \forall i, \forall \boldsymbol{y},\ \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) \geq \gamma \text{cost}(\boldsymbol{y}, \boldsymbol{y}_i)
\end{aligned}$$

Max-Margin Markov Networks

- Starting point: multiclass SVM (Crammer and Singer, 2001)

$$
\begin{aligned}
&\max_{\mathbf{w}} \gamma \\
&\quad \text{s.t.} \quad \|\mathbf{w}\| \leq 1 \\
&\qquad\quad \forall i, \forall \boldsymbol{y},\ \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) \geq \gamma \mathrm{cost}(\boldsymbol{y}, \boldsymbol{y}_i)
\end{aligned}
$$

Max-Margin Markov Networks

- Standard transformation to get rid of explicit mention of γ, plus slack variables in case the constraints cannot be met:

$$\min_{\mathbf{w}} \frac{C}{2}\|\mathbf{w}\|_2^2 + \sum_{i=1}^{N} \xi_i$$
$$\text{s.t.} \quad \forall i, \forall \boldsymbol{y},\ \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) \geq \mathrm{cost}(\boldsymbol{y}, \boldsymbol{y}_i) - \xi_i$$

- Notice:

$$\begin{aligned} \forall i, \forall \boldsymbol{y},\ \xi_i &\geq -\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) + \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) + \mathrm{cost}(\boldsymbol{y}, \boldsymbol{y}_i) \\ \forall i,\ \xi_i &\geq \max_{\boldsymbol{y}} -\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) + \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) + \mathrm{cost}(\boldsymbol{y}, \boldsymbol{y}_i) \end{aligned}$$

Max-Margin Markov Networks

- Having solved for the slack variables, we can plug in; we now have an unconstrained problem:

$$\min_{\mathbf{w}} \frac{C}{2}\|\mathbf{w}\|_2^2 + \sum_{i=1}^{N} -\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) + \max_{\boldsymbol{y}} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) + \mathrm{cost}(\boldsymbol{y}, \boldsymbol{y}_i)$$

- Ratliff, Bagnell, and Zinkevich (2007): subgradient descent (or stochastic version) – much, much simpler approach to optimizing this function.
  - And more perceptron-like!

$$-g_j(\boldsymbol{x}, \boldsymbol{y}) + g_j(\boldsymbol{x}, \mathrm{cost\_augmented\_decode}(\mathbf{w}, \boldsymbol{x}))$$

Structured Hinge Loss

- Small change to the perceptron loss:

$$L(\mathbf{w}, \boldsymbol{x}, \boldsymbol{y}) \quad = \quad -\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y})$$

- Resulting subgradient:

$$-g_j(\boldsymbol{x}, \boldsymbol{y}) + g_j(\boldsymbol{x}, \mathrm{cost\_augmented\_decode}(\mathbf{w}, \boldsymbol{x}))$$

  – Rather than merely decoding, find a candidate y' that is both high-scoring and *dangerous*.

Cost-Augmented Decoding

$$\begin{aligned}\mathrm{decode}(\mathbf{w}, \boldsymbol{x}) &= \arg\max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') \\ \mathrm{cost\_augmented\_decode}(\mathbf{w}, \boldsymbol{x}, \boldsymbol{y}) &= \arg\max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y})\end{aligned}$$

- Efficient decoding is possible when the features factor locally:

$$\mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) = \sum_p \mathbf{f}(\boldsymbol{x}, \mathrm{part}_p(\boldsymbol{y}))$$

- Efficient cost-augmented decoding requires that the cost function break into parts the same way:

$$\mathrm{cost}(\boldsymbol{y}', \boldsymbol{y}) = \sum_p \mathrm{local\_cost}(\mathrm{part}_p(\boldsymbol{y}'), \boldsymbol{y})$$

An Exercise

- If the features are such that we can use the Viterbi algorithm for decoding, what are some cost functions we could inside an efficient cost-augmented decoding algorithm that’s a very small change to Viterbi?

Structured Hinge

- Three different lines of work all arrived at this idea, or something very close.
  - Max-margin Markov networks (Taskar, Guestrin, and Koller, 2003)
  - Structural support vector machines (Tsochantaridis, Joachims, Hoffman, and Altun, 2005)
  - Online passive-aggressive algorithms (Crammer, Keshet, Dekel, Shalev-Shwartz, and Singer, 2006)
- Important developments in optimization techniques since then!
  - I’ll highlight what I think it’s most useful to know.

Max-Margin Markov Networks

- Taskar et al. actually work through a *dual* version of the problem.
  - Primal and dual are both QPs; exponentially many constraints or variables, respectively.
- Key trick: *factored dual*.
  - Enables kernelized factors in the MN.
  - Actual algorithm is sequential minimal optimization (SMO) for SVMs, a coordinate ascent method (Platt, 1999).
- The paper includes a generalization bound that is argued to improve over the Collins perceptron.
- Experiments: handwriting recognition, text classification for hyperlinked documents.

I'm Taking Liberties

- The M3N view of the world really thinks about outputs as configurations in a Markov network.
- They assume y corresponds to a set of random variables, each of which gets a label in a finite set.
- Their cost function is Hamming cost: "how many r.v.s do I predict incorrectly?"
  - This is convenient and makes sense for their applications. But it's not as general as it could be.

Structural SVM

- Tsochantaridis et al. (2005) – extends their 2004 paper.
- Slightly different version of the loss function:

$$\min_{\mathbf{w}} \frac{C}{2}\|\mathbf{w}\|_2^2 + \sum_{i=1}^{N} \xi_i$$

$$\text{s.t.} \quad \forall i, \forall \boldsymbol{y},\ \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) \geq +1 - \frac{\xi_i}{\text{cost}(\boldsymbol{y}, \boldsymbol{y}_i)}$$

  – Alternative version of cost-augmented decoding ("slack rescaling" as opposed to Taskar et al.'s "margin rescaling")

Optimization Algorithms for SSVMs

- Taskar et al. (2003): SMO based on factored dual
- Bartlett et al. (2004) and Collins et al. (2008): exponentiated gradient
- Tsochantaridis et al. (2005): cutting plane (based on dual)

Cutting Plane

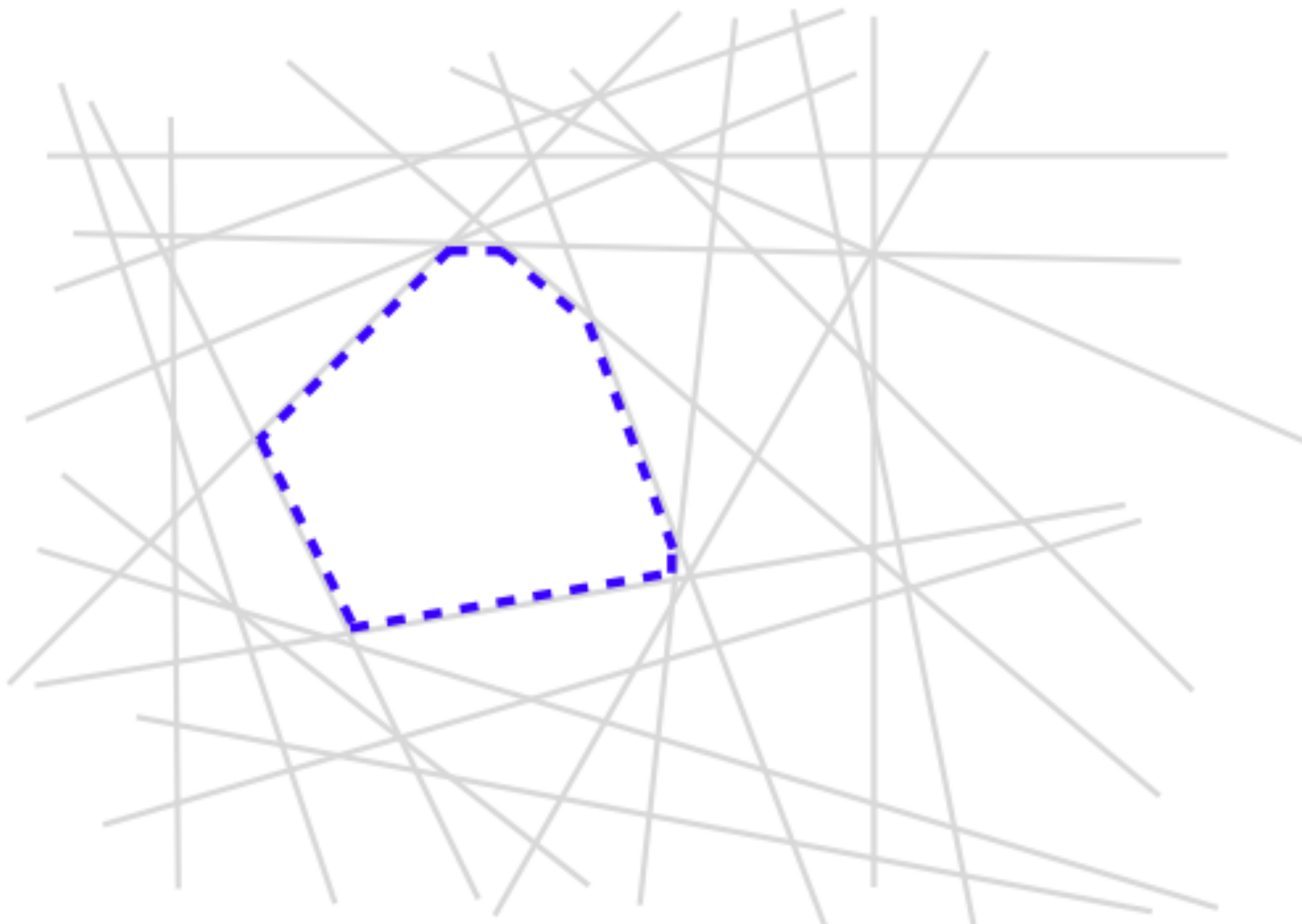
- There are exponentially many constraints!

$$\min_{\mathbf{w}} \frac{C}{2}\|\mathbf{w}\|_2^2 + \sum_{i=1}^{N} \xi_i$$

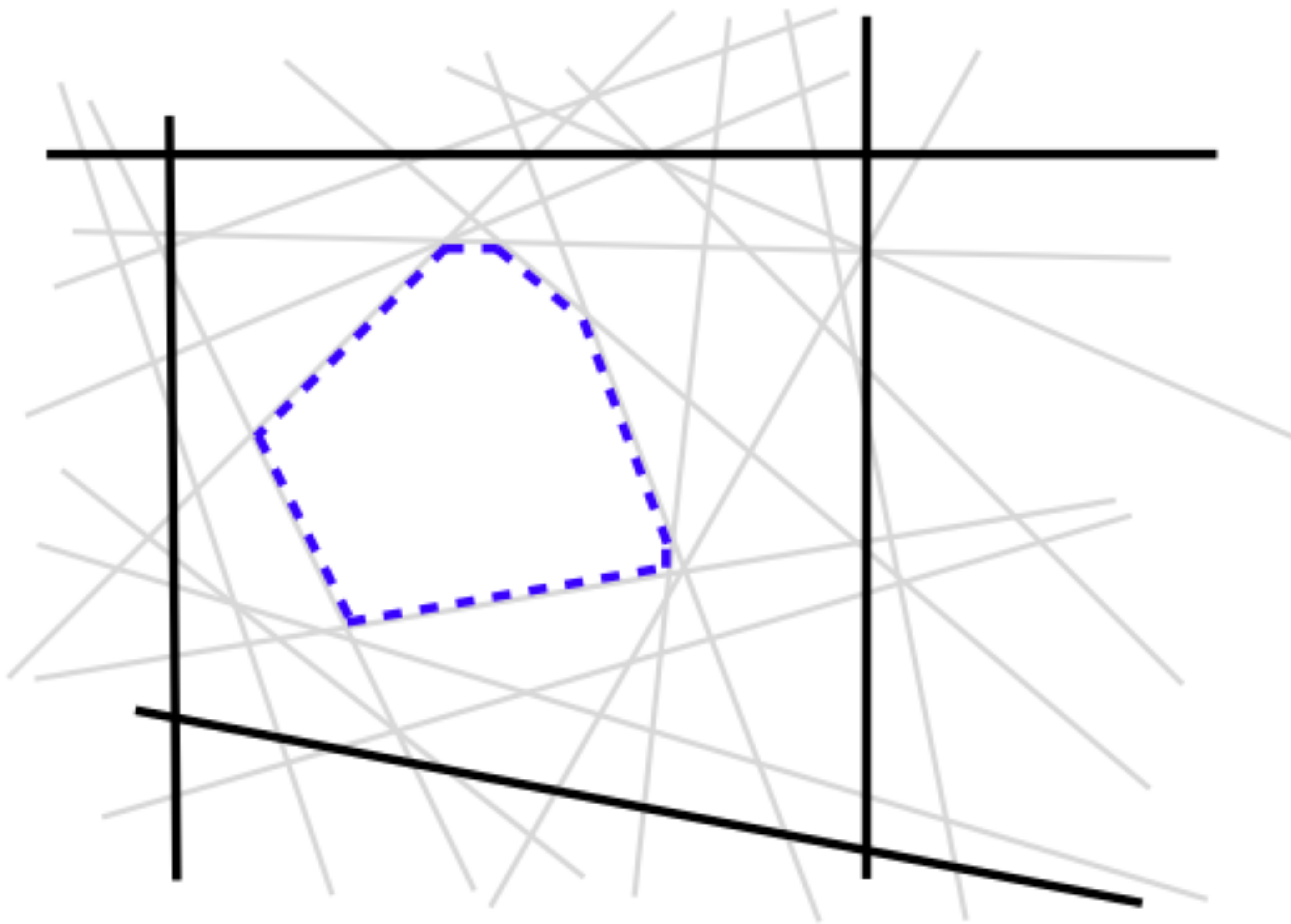
$$\text{s.t.} \quad \forall i, \forall \boldsymbol{y}, \; \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}_i) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}_i, \boldsymbol{y}) \geq \text{cost}(\boldsymbol{y}, \boldsymbol{y}_i) - \xi_i$$

- Instead of enumerating them all, let's dynamically add constraints as needed
- Iterate: solve a relaxation with a subset of constraints, then add most violated constraint

Cutting Plane



Cutting Plane



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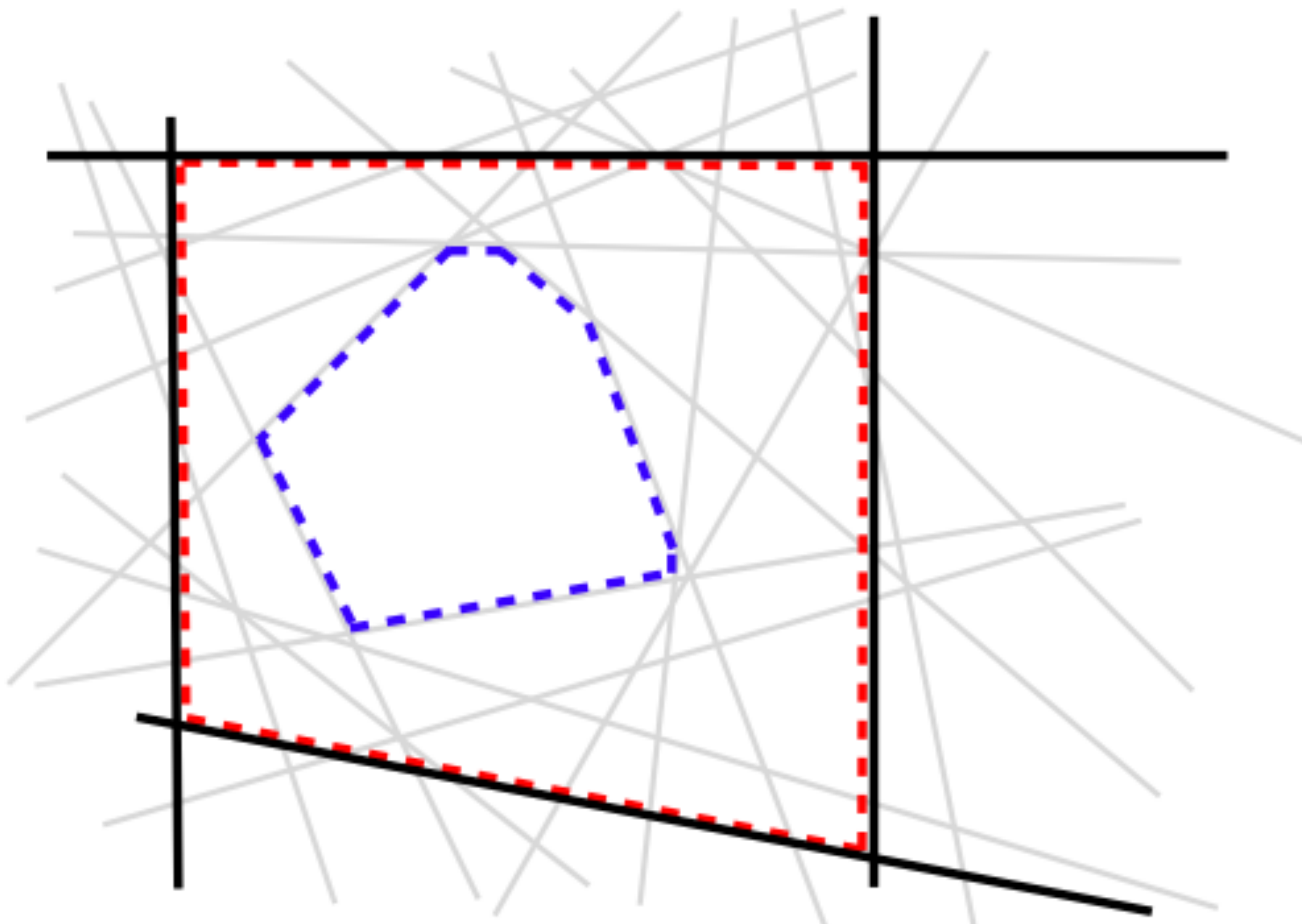

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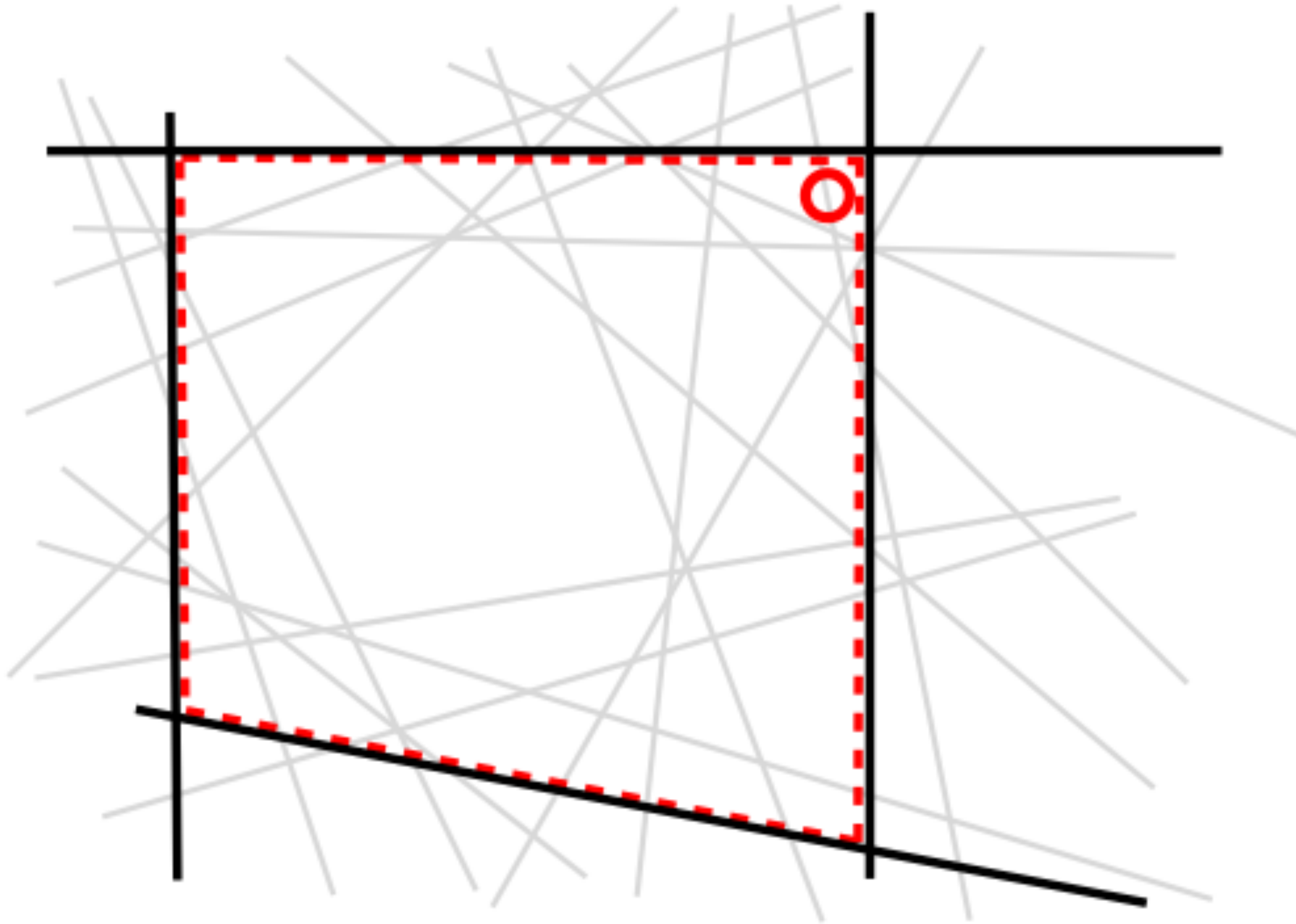

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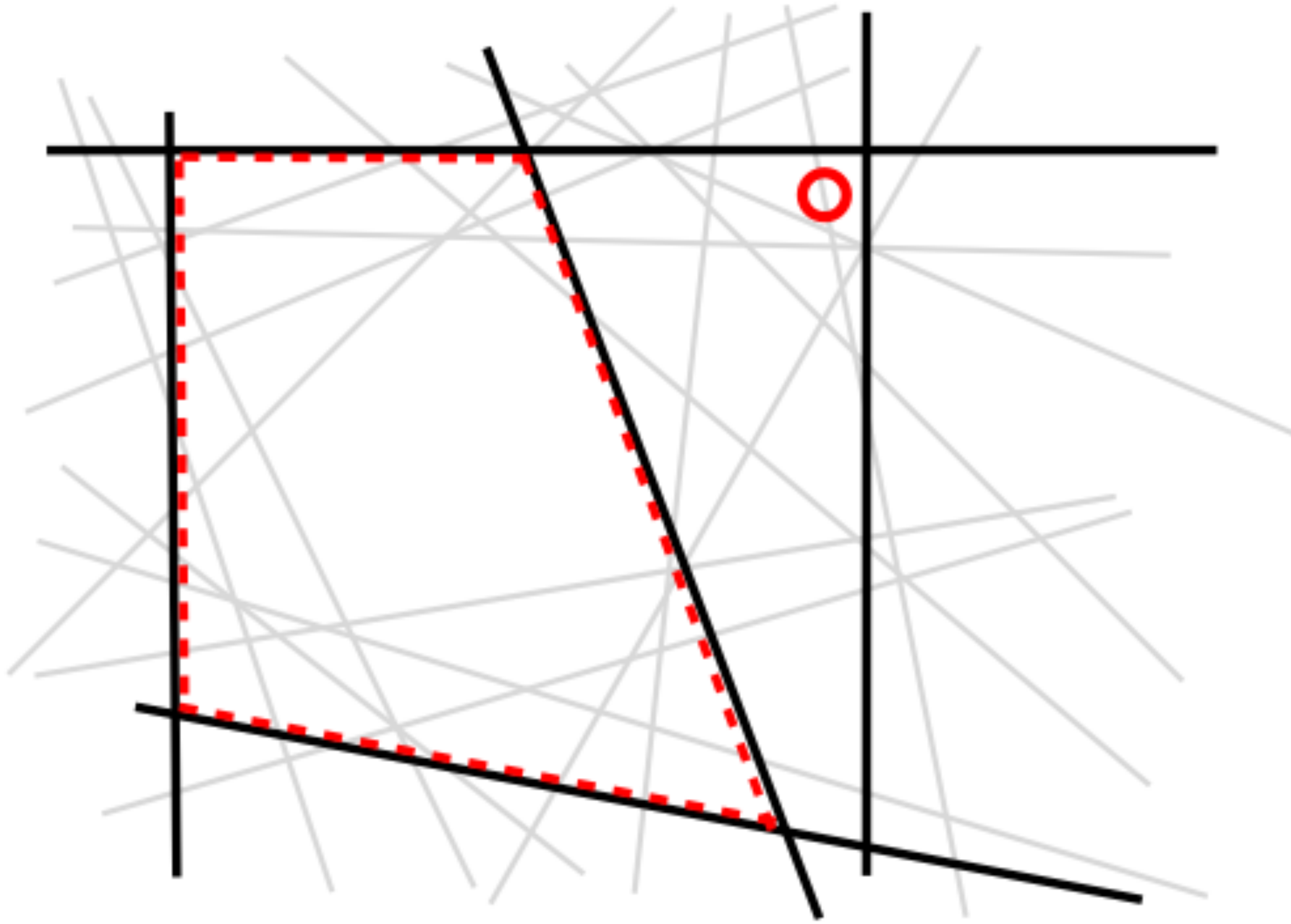

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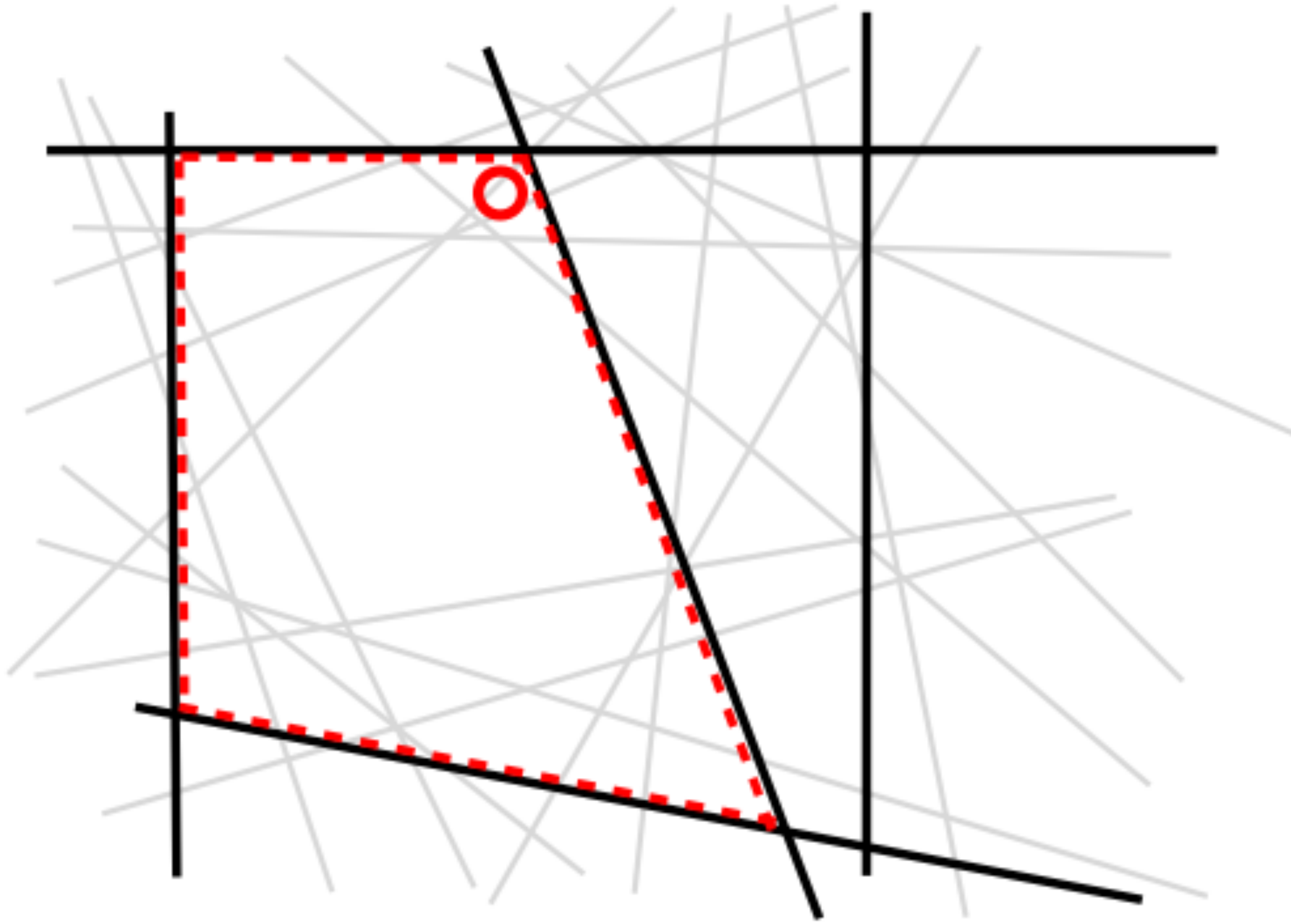

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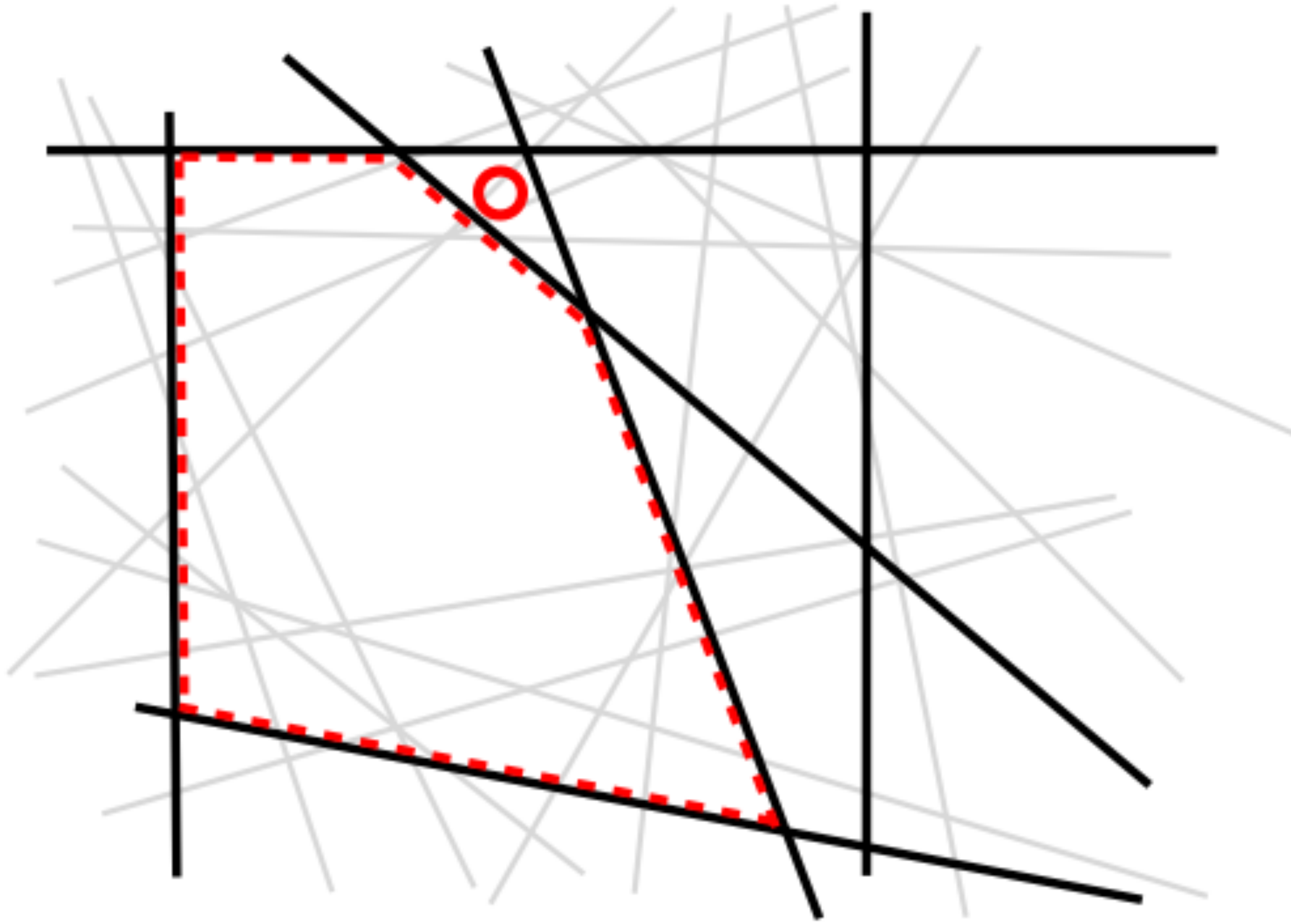

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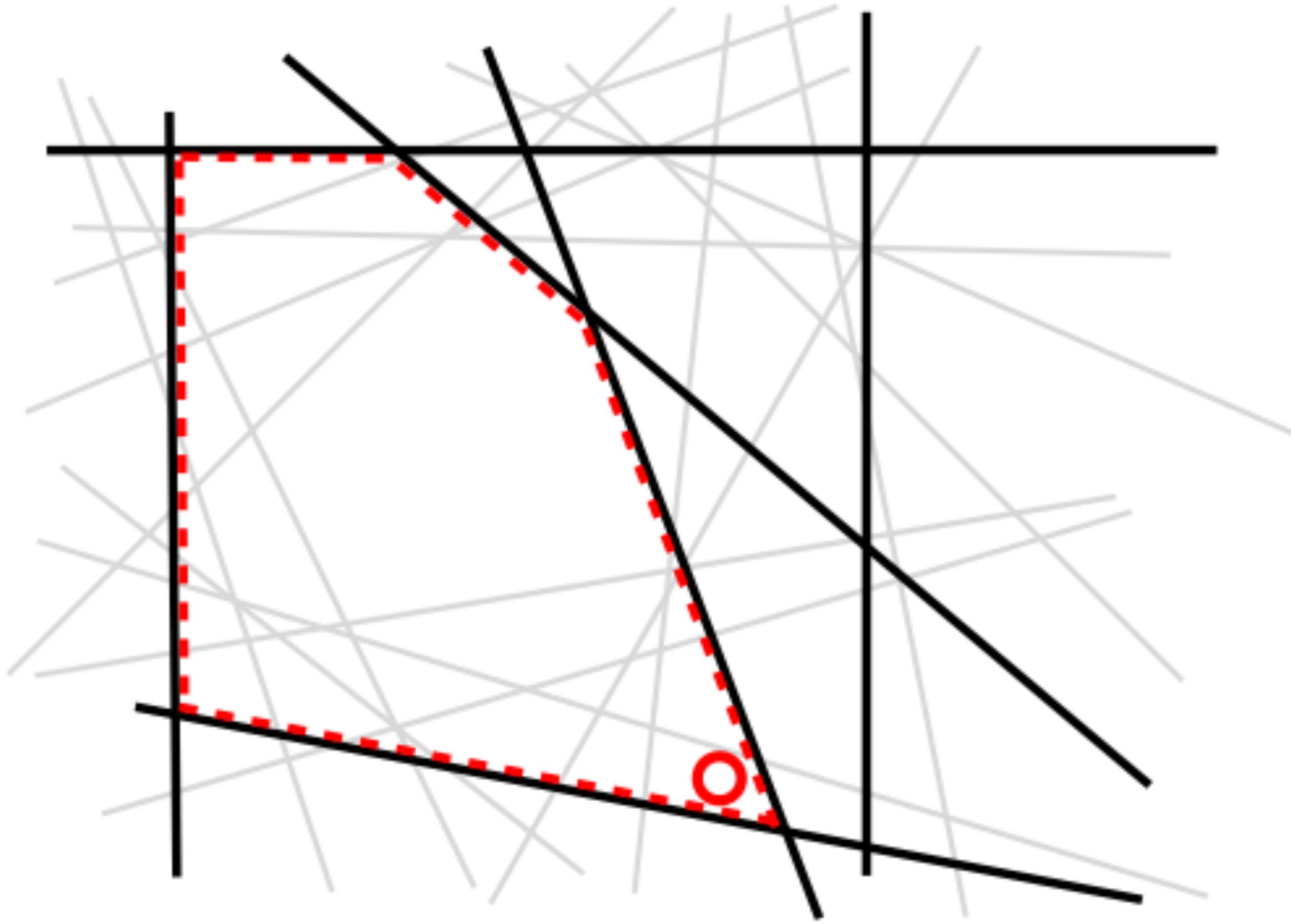

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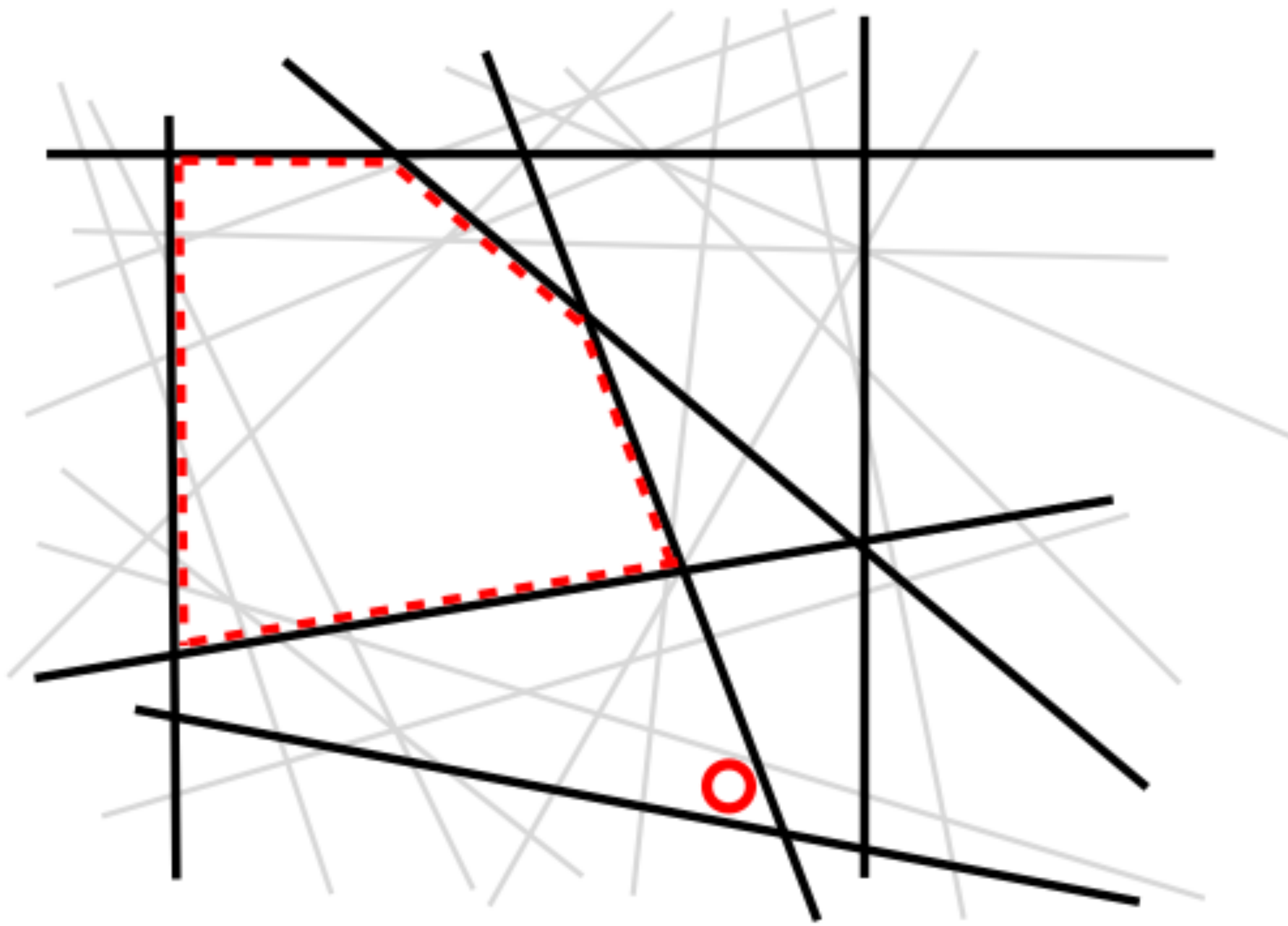

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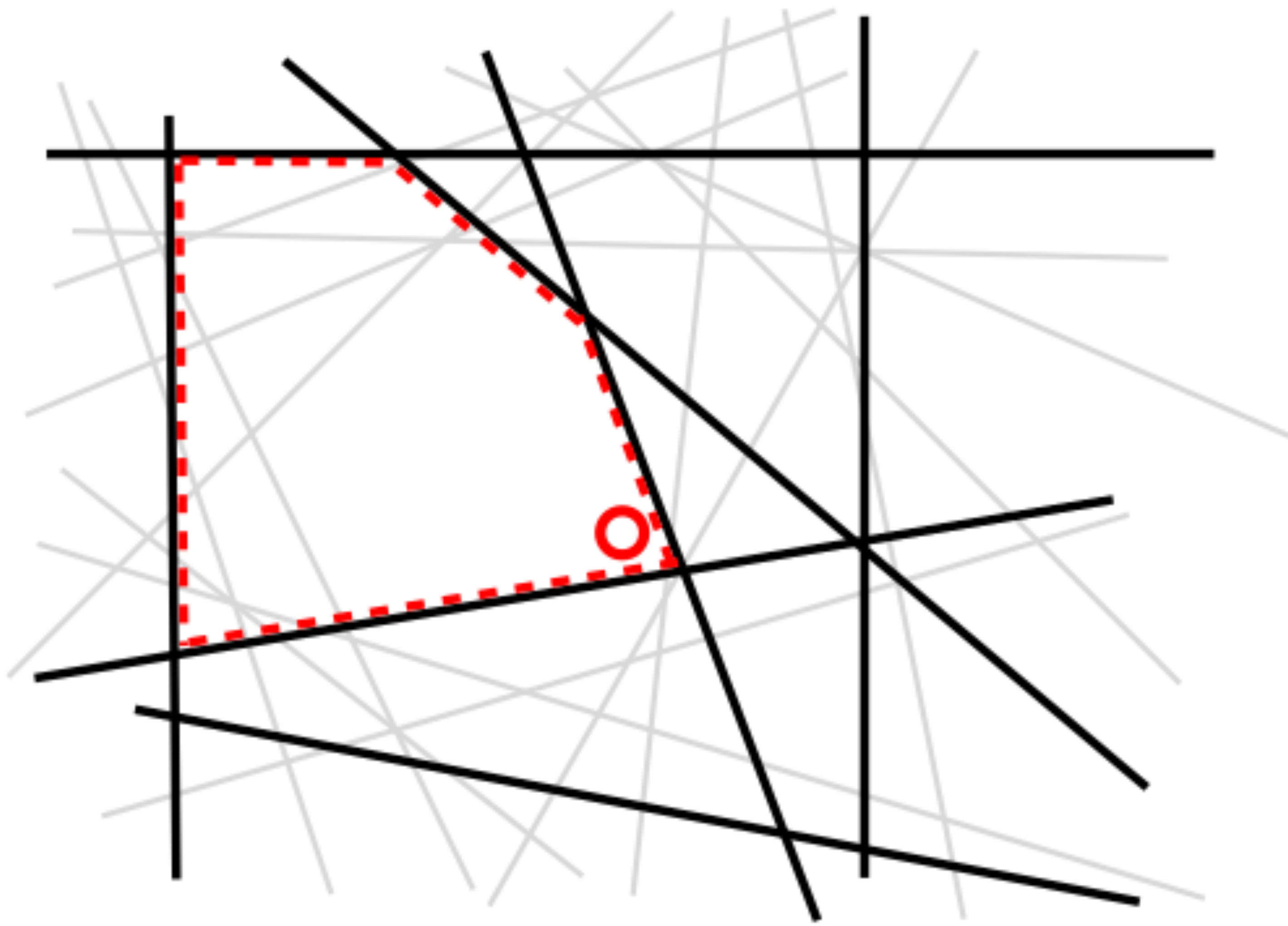

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- Bartlett et al. (2004) and Collins et al. (2008): exponentiated gradient
- Tsochantaridis et al. (2005): cutting plane (based on dual)
- Taskar et al. (2005): dual extragradient

Easiest to use, in my opinion:

- Ratliff et al. (2006): (stochastic) subgradient descent
- Crammer et al. (2006): online "passive-aggressive" algorithms

Stochastic Subgradient Descent

- Unconstrained primal objective:

$$L(\mathbf{w}, \boldsymbol{x}, \boldsymbol{y}) \quad = \quad -\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y})$$

- Resulting subgradient:

$$-g_j(\boldsymbol{x}, \boldsymbol{y}) + g_j(\boldsymbol{x}, \mathrm{cost\_augmented\_decode}(\mathbf{w}, \boldsymbol{x}))$$

– Only requires loss-augmented decode! No need for marginals / summing (cf. CRF)

"Passive Aggressive" Learners

- Starting point is the perceptron, and the focus is on the step size.
- In NLP, people often use a specific instance called "1-best MIRA" (margin infused relaxation algorithm).
  - Sometimes with regular decoding, sometimes cost-augmented decoding.
- I do not understand the name (kind of I do)

Passive-Aggressive Update in a Nutshell ("1-best MIRA")

- Given x (and y), perform decoding (or cost-augmented decoding) to obtain y'.
- To get the updated weights, solve:

$$\min_{\mathbf{w}'} \|\mathbf{w}' - \mathbf{w}\|_2^2$$
$$\text{s.t.} \quad \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') \geq \text{cost}(\boldsymbol{y}', \boldsymbol{y})$$

- Closed form solution!
  - Essentially, a subgradient update with a closed-form step size.

Perceptron and PA

- The PA papers (e.g., Crammer et al., 2006) take a procedural view of online learning and prove convergence and regret-style bounds.
- An alternative view, described by Martins et al. (2010), derives the same updates via dual coordinate ascent.
  - Confusing name: it doesn't work in the dual!
  - More general: applies to many other loss functions, so you can get a closed-form step size for perceptron and CRFs.
  - Assumes L2 regularization; role of regularization constant C is very clear in the form of the update.

Dual Coordinate Ascent Update

$$\mathbf{w} \quad \leftarrow \quad \mathbf{w} - \underbrace{\min\left\{\frac{1}{C}, \frac{L(\mathbf{w}, \boldsymbol{x}, \boldsymbol{y})}{\|\nabla_{\mathbf{w}} L(\mathbf{w}, \boldsymbol{x}, \boldsymbol{y})\|_2^2}\right\}}_{\text{step size}} \underbrace{\nabla_{\mathbf{w}} L(\mathbf{w}, \boldsymbol{x}, \boldsymbol{y})}_{\text{subgradient}}$$

- Assumes L2 regularization.
- 1-best MIRA is a special case with structured hinge loss.
- Can get regularization into perceptron this way (use perceptron loss).
- Can get closed-form step size for CRF stochastic GD.

Hinge Loss and Log Loss

- Hinge loss (M3N):

$$-\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y})$$

- Log loss (CRF):

$$-\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \log \sum_{\boldsymbol{y}'} \exp \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}')$$

Aside: Probabilities *and* Cost?

- Hinge loss (M3N):

$$-\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y})$$

- Log loss (CRF):

$$-\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \log \sum_{\boldsymbol{y}'} \exp \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}')$$

- “Softmax margin” (Gimpel and Smith, 2010):

$$-\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) + \log \sum_{\boldsymbol{y}'} \exp \left(\mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y})\right)$$

Loss Functions You Know

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Cost	$\mathrm{cost}(\mathrm{decode}(\mathbf{w}, \boldsymbol{x}), \boldsymbol{y})$
Expected cost, a.k.a. “risk”	$\mathbb{E}_{p(\boldsymbol{Y} \mid \boldsymbol{x}, \mathbf{w})}[\mathrm{cost}(\boldsymbol{Y}, \boldsymbol{y})]$
Perceptron loss	$\max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$
Hinge (margin rescaling version)	$\max_{\boldsymbol{y}'} \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}') + \mathrm{cost}(\boldsymbol{y}', \boldsymbol{y}) - \mathbf{w}^\top \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$

On Regularization

- In principle, this choice is orthogonal to the loss function.
- L2 is the most common starting place.
- L1 and other sparsity-inducing regularizers have some nice properties, but they can make optimization more complicated

Does this matter?

Practical Advice

- Features still more important than the loss function.
  - But general, easy-to-implement algorithms are quite useful!
- Perceptron is easiest to implement.
- CRFs and SSVMs usually do better.
- If the cost function factors locally, I recommend using a hinge loss and stochastic subgradient descent.
- Tune the regularization constant.
  - Never on the test data.